ON THE NUMBER OF TERMS IN THE MIDDLE OF ALMOST SPLIT SEQUENCES OVER CYCLE-FINITE ARTIN ALGEBRAS

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Abstract. We prove that the number of terms in the middle of an almost split sequence in the module category of a cycle-finite artin algebra is bounded by 5.

1. Introduction and the main result

Throughout this paper, by an algebra is meant an artin algebra over a fixed commutative artin ring K, which we moreover assume (without loss of generality) to be basic and indecomposable. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by the indecomposable modules, by Γ_A the Auslander-Reiten quiver of A, and by τ_A and τ_A^{-1} the Auslander-Reiten translations $D\mathrm{Tr}$ and $\mathrm{Tr}D$, respectively. We do not distinguish between a module in $\operatorname{ind} A$ and the vertex of Γ_A corresponding to it. The Jacobson radical rad_A of $\operatorname{mod} A$ is the ideal generated by all nonisomorphisms between modules in $\operatorname{ind} A$, and the infinite radical $\operatorname{rad}_A^\infty$ of $\operatorname{mod} A$ is the intersection of all powers rad_A^i, $i \geq 1$, of rad_A. By a theorem of M. Auslander [4], $\operatorname{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, $\operatorname{ind} A$ admits only a finite number of pairwise nonisomorphic modules. On the other hand, if A is of infinite representation type then $(\operatorname{rad}_A^\infty)^2 \neq 0$, by a theorem proved in [11].

A prominent role in the representation theory of algebras is played by almost split sequences introduced by M. Auslander and I. Reiten in [5] (see [7] for general theory and applications). For an algebra A and a nonprojective module X in $\operatorname{ind} A$, there is an almost split sequence

$$0 \to \tau_A X \to Y \to X \to 0,$$

with $\tau_A X$ a noninjective module in $\operatorname{ind} A$ called the Auslander-Reiten translation of X. Then we may associate to X the numerical invariant $\alpha(X)$

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being the number of summands in a decomposition $Y = Y_1 \oplus \ldots \oplus Y_r$ of Y into a direct sum of modules in $\operatorname{ind} A$. Then $\alpha(X)$ measures the complication of homomorphisms in $\operatorname{mod} A$ with domain $\tau_A X$ and codomain X. Therefore, it is interesting to study the relation between an algebra A and the values $\alpha(X)$ for all modules X in $\operatorname{ind} A$ (we refer to [6], [8], [10], [21], [25], [28], [29], [31], [45], [46] for some results in this direction). In particular, it has been proved by R. Bautista and S. Brenner in [8] that, if A is of finite representation type and X a nonprojective module in $\operatorname{ind} A$, then $\alpha(X) \leq 4$, and if $\alpha(X) = 4$ then the middle term Y of an almost split sequence in $\operatorname{mod} A$ with the right term X admits an indecomposable projective-injective direct summand P, and hence $X = P/\operatorname{soc}(P)$. In [25] S. Liu generalized this result by showing that the same holds for any nonprojective module X in $\operatorname{ind} A$ over an algebra A provided $\tau_A X$ has a projective predecessor and X has an injective successor in Γ_A, as well as for X lying on an oriented cycle in Γ_A (see also [21]). It has been conjectured by S. Brenner that $\alpha(X) \leq 5$ for any nonprojective module X in $\operatorname{ind} A$ for an arbitrary tame finite dimensional algebra A over an algebraically closed field K. In fact, it is expected that this also holds for nonprojective indecomposable modules over arbitrary generically tame (in the sense of [12], [13]) artin algebras.

The main aim of this paper is to prove the following theorem which gives the affirmative answer for the above conjecture in the case of cycle-finite artin algebras.

Theorem. *Let A be a cycle-finite algebra and X be a nonprojective module in* $\operatorname{ind} A$, *and*

$$0 \to \tau_A X \to Y \to X \to 0$$

be the associated almost split sequence in $\operatorname{mod} A$. *The following statements hold.*

(i) $\alpha(X) \leq 5$.
(ii) *If $\alpha(X) = 5$ then Y admits an indecomposable projective-injective direct summand P, and hence $X = P/\operatorname{soc}(P)$.*

We would like to mention that, for finite dimensional cycle-finite algebras A over an algebraically closed field K, the theorem was proved by J. A. de la Peña and M. Takane [29, Theorem 3], by application of spectral properties of Coxeter transformations of algebras and results established in [25].

Let A be an algebra. Recall that a cycle in $\operatorname{ind} A$ is a sequence

$$X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{r-1} \xrightarrow{f_r} X_r = X_0$$

of nonzero nonisomorphisms in $\operatorname{ind} A$ [35], and such a cycle is said to be finite if the homomorphisms $f_1, \ldots, f_r$ do not belong to $\operatorname{rad}_A^\infty$. Then, following [3], [40], an algebra A is said to be cycle-finite if all cycles in $\operatorname{ind} A$ are finite. The class of cycle-finite algebras contains the following distinguished classes of algebras: the algebras of finite representation type, the hereditary algebras of Euclidean type [14], [15], the tame tilted algebras [17], [19],

[35], the tame double tilted algebras [32], the tame generalized double tilted algebras [33], the tubular algebras [35], the iterated tubular algebras [30], the tame quasi-tilted algebras [22], [43], the tame generalized multicoil algebras [26], the algebras with cycle-finite derived categories [2], and the strongly simply connected algebras of polynomial growth [41]. On the other hand, frequently an algebra A admits a Galois covering $R \to R/G = A$, where R is a cycle-finite locally bounded category and G is an admissible group of automorphisms of R, which allows to reduce the representation theory of A to the representation theory of cycle-finite algebras being finite convex subcategories of R (see [16], [28], [42] for some general results). For example, every finite dimensional selfinjective algebra of polynomial growth over an algebraically closed field admits a canonical standard form $\overline{A}$ (geometric socle deformation of A) such that $\overline{A}$ has a Galois covering $R \to R/G = \overline{A}$, where R is a cycle-finite selfinjective locally bounded category and G is an admissible infinite cyclic group of automorphisms of R, the Auslander-Reiten quiver $\Gamma_{\overline{A}}$ of $\overline{A}$ is the orbit quiver Γ_R/G of Γ_R, and the stable Auslander-Reiten quivers of A and $\overline{A}$ are isomorphic (see [36] and [44]). Recall also that, a module X in $\operatorname{ind} A$ which does not lie on a cycle in $\operatorname{ind} A$ is called directing, and its support algebra is a tilted algebra, by a result of C. M. Ringel [35]. Moreover, it has been proved independently by L. G. Peng - J. Xiao [27] and A. Skowroński [38] that the Auslander-Reiten quiver Γ_A of an algebra A admits at most finitely many τ_A-orbits containing directing modules.

2. Preliminary results

Let H be an indecomposable hereditary algebra and Q_H the valued quiver of H. Recall that the vertices of Q_H are the numbers $1, 2, \ldots, n$ corresponding to a complete set $S_1, S_2, \ldots, S_n$ of pairwise nonisomorphic simple modules in $\operatorname{mod} H$ and there is an arrow from i to j in Q_H if $\operatorname{Ext}^1_H(S_i, S_j) \neq 0$, and then to this arrow is assigned the valuation $(\dim_{\operatorname{End}_H(S_j)} \operatorname{Ext}^1_H(S_i, S_j)$, $\dim_{\operatorname{End}_H(S_i)} \operatorname{Ext}^1_H(S_i, S_j))$. Recall also that the Auslander-Reiten quiver Γ_H of H has a disjoint union decomposition of the form

$$\Gamma_H = \mathcal{P}(H) \vee \mathcal{R}(H) \vee \mathcal{Q}(H),$$

where $\mathcal{P}(H)$ is the preprojective component containing all indecomposable projective H-modules, $\mathcal{Q}(H)$ is the preinjective component containing all indecomposable injective H-modules, and $\mathcal{R}(H)$ is the family of all regular components of Γ_H. More precisely, we have:

- if Q_H is a Dynkin quiver, then $\mathcal{R}(H)$ is empty and $\mathcal{P}(H) = \mathcal{Q}(H)$;
- if Q_H is a Euclidean quiver, then $\mathcal{P}(H) \cong (-\mathbb{N})Q_H^{\mathrm{op}}$, $\mathcal{Q}(H) \cong \mathbb{N}Q_H^{\mathrm{op}}$ and $\mathcal{R}(H)$ is a strongly separating infinite family of stable tubes;
- if Q_H is a wild quiver, then $\mathcal{P}(H) \cong (-\mathbb{N})Q_H^{\mathrm{op}}$, $\mathcal{Q}(H) \cong \mathbb{N}Q_H^{\mathrm{op}}$ and $\mathcal{R}(H)$ is an infinite family of components of type $\mathbb{Z}\mathbb{A}_\infty$.

Let T be a tilting module in $\operatorname{mod} H$ and $B = \operatorname{End}_H(T)$ the associated tilted algebra. Then the tilting H-module T determines the torsion pair $(\mathcal{F}(T), \mathcal{T}(T))$ in $\operatorname{mod} H$, with the torsion-free part $\mathcal{F}(T) = \{X \in \operatorname{mod} H | \operatorname{Hom}_H(T, X) = 0\}$ and the torsion part $\mathcal{T}(T) = \{X \in \operatorname{mod} H | \operatorname{Ext}^1_H(T, X) = 0\}$, and the splitting torsion pair $(\mathcal{Y}(T), \mathcal{X}(T))$ in $\operatorname{mod} B$, with the torsion-free part $\mathcal{Y}(T) = \{Y \in \operatorname{mod} B | \operatorname{Tor}^B_1(Y, T) = 0\}$ and the torsion part $\mathcal{X}(T) = \{Y \in \operatorname{mod} B | Y \otimes_B T = 0\}$. Then, by the Brenner-Butler theorem, the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} B$ induces an equivalence of $\mathcal{T}(T)$ with $\mathcal{Y}(T)$, and the functor $\operatorname{Ext}^1_H(T, -) : \operatorname{mod} H \to \operatorname{mod} B$ induces an equivalence of $\mathcal{F}(T)$ with $\mathcal{X}(T)$ (see [9], [17]). Further, the images $\operatorname{Hom}_H(T, I)$ of the indecomposable injective modules I in $\operatorname{mod} H$ via the functor $\operatorname{Hom}_H(T, -)$ belong to one component $\mathcal{C}_T$ of Γ_B, called the connecting component of Γ_B determined by T, and form a faithful section Δ_T of $\mathcal{C}_T$, with Δ_T the opposite valued quiver Q_H^{op} of Q_H. Recall that a full connected valued subquiver Σ of a component $\mathcal{C}$ of Γ_B is called a section (see [1, (VIII.1)]) if Σ has no oriented cycles, is convex in $\mathcal{C}$, and intersects each τ_B-orbit of $\mathcal{C}$ exactly once. Moreover, the section Σ is faithful provided the direct sum of all modules lying on Σ is a faithful B-module. The section Δ_T of the connecting component $\mathcal{C}_T$ of Γ_B has the distinguished property: it connects the torsion-free part $\mathcal{Y}(T)$ with the torsion part $\mathcal{X}(T)$, because every predecessor in $\operatorname{ind} B$ of a module $\operatorname{Hom}_H(T, I)$ from Δ_T lies in $\mathcal{Y}(T)$ and every successor of $\tau_B^- \operatorname{Hom}_H(T, I)$ in $\operatorname{ind} B$ lies in $\mathcal{X}(T)$. We note that, by a result proved in [24] and [37], an algebra A is a tilted algebra if and only if Γ_A admits a component $\mathcal{C}$ with a faithful section Δ such that $\operatorname{Hom}_A(X, \tau_A Y) = 0$ for all modules X and Y from Δ. We refer also to [18] for another characterization of tilted algebras involving short chains of modules.

The following proposition is a well-known fact.

Proposition 2.1. *Let H be a hereditary algebra of Euclidean type. Then, for any nonprojective indecomposable module X in* $\operatorname{mod} H$, *we have $\alpha(X) \leq 4$.*

An essential role in the proof of the main theorem will be played by the following theorem.

Theorem 2.2. *Let A be a cycle-finite algebra, $\mathcal{C}$ a component of Γ_A, and $\mathcal{D}$ be an acyclic left stable full translation subquiver of $\mathcal{C}$ which is closed under predecessors. Then there exists a hereditary algebra H of Euclidean type and a tilting module T in* $\operatorname{mod} H$ *without nonzero preinjective direct summands such that for the associated tilted algebra $B = \operatorname{End}_H(T)$ the following statements hold.*

(i) *B is a quotient algebra of A.*
(ii) *The torsion-free part $\mathcal{Y}(T) \cap \mathcal{C}_T$ of the connecting component $\mathcal{C}_T$ of Γ_B determined by T is a full translation subquiver of $\mathcal{D}$ which is closed under predecessors in $\mathcal{C}$.*
(iii) *For any indecomposable module N in $\mathcal{D}$, we have $\alpha(N) \leq 4$.*

Proof. Since A is a cycle-finite algebra, every acyclic module X in Γ_A is a directing module in $\operatorname{ind} A$. Hence $\mathcal{D}$ consists entirely of directing modules. Moreover, it follows from [27, Theorem 2.7] and [38, Corollary 2], that $\mathcal{D}$ has only finitely many τ_A-orbits. Then, applying [23, Theorem 3.4], we conclude that there is a finite acyclic valued quiver Δ such that $\mathcal{D}$ contains a full translation subquiver Γ which is closed under predecessors in $\mathcal{C}$ and is isomorphic to the translation quiver $\mathbb{N}\Delta$. Therefore, we may choose in Γ a finite acyclic convex subquiver Δ such that Γ consists of the modules $\tau_A^m X$ with $m \geq 0$ and X indecomposable modules lying on Δ. Let M be the direct sum of all indecomposable modules in $\mathcal{C}$ lying on the chosen quiver Δ. Let I be the annihilator $\operatorname{ann}_A(M) = \{a \in A \mid Ma = 0\}$ of M in A, and $B = A/I$ the associated quotient algebra. Then $I = \operatorname{ann}_A(\Gamma)$ (see [37, Lemma 3]) and consequently Γ consists of indecomposable B-modules. Clearly, B is a cycle-finite algebra, as a quotient algebra of A. Now, using the fact that $\Gamma \subseteq \mathbb{N}\Delta$ and consists of directing B-modules, we conclude that $\operatorname{rad}_B^{\infty}(M, M) = 0$ and $\operatorname{Hom}_B(M, \tau_B M) = 0$. Then, applying [39, Lemma 3.4], we conclude that $H = \operatorname{End}_B(M)$ is a hereditary algebra and the quiver Q_H of H is the dual valued quiver Δ^{op} of Δ. Further, since M is a faithful B-module with $\operatorname{Hom}_B(M, \tau_B M) = 0$, we conclude that $\operatorname{pd}_B M \leq 1$ and $\operatorname{Ext}_B^1(M, M) \cong D\overline{\operatorname{Hom}}_B(M, \tau_B M) = 0$ (see [1, Lemma VIII.5.1 and Theorem IV.2.13]). Moreover, it follows from definition of M that, for any module Z in $\operatorname{ind} B$ with $\operatorname{Hom}_B(M, Z) \neq 0$ and not on Δ, we have $\operatorname{Hom}_B(\tau_B^{-1} M, Z) \neq 0$. Since M is a faithful module in $\operatorname{mod} B$ there is a monomorphism $B \to M^s$ for some positive integer s. Then $\operatorname{rad}_B^{\infty}(M, M) = 0$ implies $\operatorname{Hom}_B(\tau_B^{-1} M, B) = 0$, and consequently $\operatorname{id}_B M \leq 1$. Applying now [34, Lemma 1.6] we conclude that M is a tilting B-module. Further, applying the Brenner-Butler theorem (see [1, Theorem VI.3.8]), we conclude that M is a tilting module in $\operatorname{mod} H^{\mathrm{op}}$ and $B \cong \operatorname{End}_{H^{\mathrm{op}}}(M)$. Since H is a hereditary algebra, $T = D(M)$ is a tilting module in $\operatorname{mod} H$ with $B \cong \operatorname{End}_H(T)$, and consequently B is a tilted algebra of type $Q_H = \Delta^{\mathrm{op}}$. Moreover, the translation quiver Γ is the torsion-free part $\mathcal{Y}(T) \cap \mathcal{C}_T$ of the connecting component $\mathcal{C}_T$ of Γ_B determined by the tilting H-module T (see [1, Theorem VIII.5.6]). Observe that then $\mathcal{Y}(T) \cap \mathcal{C}_T$ is the image $\operatorname{Hom}_H(T, Q(H))$ of the preinjective component $Q(H)$ of Γ_H via the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} B$. In particular, we conclude that H is of infinite representation type (Q_H is not a Dynkin quiver) and $\mathcal{C}_T$ does not contain a projective module, and hence T is without nonzero preinjective direct summands (see [1, Proposition VIII.4.1]). Finally, we prove that $Q_H = \Delta^{\mathrm{op}}$ is a Euclidean quiver. Suppose that Q_H is a wild quiver. Since T has no nonzero preinjective direct summands, it follows from [20] that Γ_B admits an acyclic component Σ with infinitely many τ_B-orbits, with the stable part $\mathbb{Z}\mathbb{A}_{\infty}$, contained entirely in the torsion-free part $\mathcal{Y}(T)$ of $\operatorname{mod} B$. Since B is a cycle-finite algebra, Σ consists of directing B-modules, and hence Γ_B contains infinitely many τ_B-orbits containing directing modules, a contradiction. Therefore, Q_H is a Euclidean quiver and B is a tilted algebra

of Euclidean type $Q_H = \Delta^{\rm op}$. This finishes proof of the statements (i) and (ii).

In order to prove (iii), consider a module N in $\mathcal{D}$ and an almost split sequence

$$0 \to \tau_A N \to E \to N \to 0$$

in $\operatorname{mod} A$ with the right term N. Since $\mathcal{D}$ is left stable and closed under predecessors in $\mathcal{C}$, we have in $\operatorname{mod} A$ almost split sequences

$$0 \to \tau_A^{m+1} N \to \tau_A^m E \to \tau_A^m N \to 0$$

for all nonnegative integers m. In particular, there exists a positive integer n such that

$$0 \to \tau_A^{n+1} N \to \tau_A^n E \to \tau_A^n N \to 0$$

is an exact sequence in the additive category $\operatorname{add}(\mathcal{Y}(T) \cap \mathcal{C}_T) = \operatorname{add}(\Gamma)$. Since $\mathcal{Y}(T) \cap \mathcal{C}_T = \operatorname{Hom}_H(T, Q(H))$, this exact sequence is the image via the functor $\operatorname{Hom}_H(T, -) : \operatorname{mod} H \to \operatorname{mod} B$ of an almost split sequence

$$0 \to \tau_H U \to V \to U \to 0$$

with all terms in the additive category $\operatorname{add}(Q(H))$ of $Q(H)$. Then, applying Proposition 2.1, we conclude that $\alpha(N) = \alpha(\tau_A^n N) = \alpha(\tau_B^n N) = \alpha(U) \leq 4$. □

3. Proof of Theorem

We will use the following results proved by S. Liu in [25] (Theorem 7, Proposition 8, Lemma 6 and its dual).

Theorem 3.1. *Let A be an algebra, and let*

$$0 \to \tau_A X \to \bigoplus_{i=1}^{r} Y_i \to X \to 0$$

be an almost split sequence in $\operatorname{mod} A$ *with* $Y_1, \dots, Y_r$ *from* $\operatorname{ind} A$. *Assume that one of the following conditions holds.*

(i) *$\tau_A X$ has a projective predecessor and X has an injective successor in Γ_A.*
(ii) *X lies on an oriented cycle in Γ_A.*

Then $r \leq 4$, and $r = 4$ implies that one of the modules Y_i is projective-injective, whereas the others are neither projective nor injective.

Proposition 3.2. *Let A be an algebra, and let*

$$0 \to \tau_A X \to \bigoplus_{i=1}^{r} Y_i \to X \to 0$$

be an almost split sequence in $\operatorname{mod} A$ *with $r \geq 5$ and* $Y_1, \dots, Y_r$ *from* $\operatorname{ind} A$. *Then the following statements hold.*

(i) *If there is a sectional path from $\tau_A X$ to an injective module in Γ_A, then $\tau_A X$ has no projective predecessor in Γ_A.*

(ii) *If there is a sectional path from a projective module in Γ_A to X, then X has no injective successor in Γ_A.*

We are now in position to prove the main result of the paper.

Let A be a cycle-finite algebra, and let

$$0 \to \tau_A X \to \bigoplus_{i=1}^{r} Y_i \to X \to 0$$

be an almost split sequence in $\operatorname{mod} A$ with $Y_1, \dots, Y_r$ from $\operatorname{ind} A$, and let $\mathcal{C}$ be the component of Γ_A containing X. Assume $r \geq 5$. We claim that then $r = 5$, one of the modules Y_i is projective-injective, whereas the others are neither projective nor injective.

Since $r \geq 5$, it follows from Theorem 3.1 that $\tau_A X$ has no projective predecessor nor X has no injective successor in Γ_A. Assume that $\tau_A X$ has no projective predecessor in Γ_A.

We claim that then one of the modules Y_i is projective. Suppose it is not the case. Then for any nonnegative integer m we have in $\operatorname{mod} A$ an almost split sequence

$$0 \to \tau_A^{m+1} X \to \bigoplus_{i=1}^{r} \tau_A^m Y_i \to \tau_A^m X \to 0$$

with $r \geq 5$ and $\tau_A^m Y_1, \dots, \tau_A^m Y_r$ from $\operatorname{ind} A$, because $\tau_A X$ has no projective predecessor in Γ_A. Moreover, it follows from Theorem 3.1, that $\tau_A^m X$, $m \geq 0$, are acyclic modules in Γ_A. Then it follows from [23, Theorem 3.4] that the modules $\tau_A^m X$, $m \geq 0$, belong to an acyclic left stable full translation subquiver $\mathcal{D}$ of $\mathcal{C}$ which is closed under predecessors. But then the assumption $r \geq 5$ contradicts Theorem 2.2(iii). Therefore, one of the modules Y_i, say Y_r is projective.

Observe now that the remaining modules $Y_1, \dots, Y_{r-1}$ are noninjective. Indeed, since Y_r is projective, we have $\ell(\tau_A X) < \ell(Y_r)$ and consequently $\sum_{i=1}^{r-1} \ell(Y_i) < \ell(X)$. Further, Y_r is a projective predecessor of X in Γ_A, and hence, applying Proposition 3.2(ii), we conclude that X has no injective successors in Γ_A. We claim that Y_r is injective. Indeed, if it is not the case, we have in $\operatorname{mod} A$ almost split sequences

$$0 \to \tau_A^{-m+1} X \to \bigoplus_{i=1}^{r} \tau_A^{-m} Y_i \to \tau_A^{-m} X \to 0$$

for all nonnegative integers m. Then, applying the dual of Theorem 2.2, we obtain a contradiction with $r \geq 5$. Thus Y_r is projective-injective. Observe that then the modules $Y_1, \dots, Y_{r-1}$ are nonprojective, because Y_r injective forces the inequalities $\ell(X) < \ell(Y_r)$ and $\sum_{i=1}^{r-1} \ell(Y_i) < \ell(\tau_A X)$.

Finally, since $\tau_A X$ has no projective predecessor in Γ_A, we have in $\operatorname{mod} A$ almost split sequences

$$0 \to \tau_A^{m+1} X \to \bigoplus_{i=1}^{r-1} \tau_A^m Y_i \to \tau_A^m X \to 0$$

for all positive integers m. Applying Proposition 3.2 again, we conclude (as in the first part of the proof) that $r - 1 \leq 4$, and hence $r \leq 5$. Therefore, $\alpha(X) = r = 5$, one of the modules Y_i is projective-injective, whereas the others are neither projective nor injective. Moreover, if Y_i is a projective-injective module, then $X \cong Y_i/\mathrm{soc}(Y_i)$.

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